Quantum-symmetric equivalence via Manin's universal quantum groups

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Outline

1. Background: universal quantum groups, Morita–Takeuchi theory, and quantum-symmetric equivalence;
2. quantum symmetric equivalences of Zhang twists;
3. quantum-symmetric equivalences of Artin–Schelter regular algebras;
4. quantum-symmetric equivalences of superpotential algebras.

Setup

- $\Bbbk$ is an algebraically closed base field, all algebras are over $\Bbbk$.
- Typically A will be an $\mathbb{N}$-graded algebra: $A = \bigoplus_{i \in \mathbb{N}} A_i$, and $A_i A_j \subseteq A_{i+j}$.
- When A is connected, this means $A_0 = \Bbbk$.
- Every coaction of a Hopf algebra H on A will be required to be graded, that is, it sends $A_i \to A_i \otimes H$.

Universal quantum groups

Idea: we want to study A by understanding Hopf coactions on A.

In general, many different Hopf algebras coact on A. To study coactions systematically, we use **Manin's universal quantum group**:

$$\begin{array}{ccc} A & \xrightarrow{\rho} & \underline{\mathrm{aut}}^l(A) \otimes A \\ & \searrow^{\tau} & \downarrow{\scriptstyle f \otimes \mathrm{id}} \\ & & H \otimes A. \end{array}$$

Right universal Hopf algebras $\underline{\mathrm{aut}}^r(A)$ are defined analogously.

Replacing everywhere "Hopf algebra" by "bialgebra," we define $\underline{\mathrm{end}}^l(A)$ and $\underline{\mathrm{end}}^r(A)$.

Existence of universal quantum groups

If A is a quadratic algebra, $\underline{\mathrm{end}}(A)$ has a concrete description:

- If A and B are algebras,

$$A \bullet B := \frac{\Bbbk\langle A_1 \otimes B_1 \rangle}{\left(S_{(23)}\left(R(A) \otimes R(B)\right)\right)}, \tag{1}$$

  where $S_{(23)} : A_1 \otimes A_1 \otimes B_1 \otimes B_1 \to A_1 \otimes B_1 \otimes A_1 \otimes B_1$ is the flip of the middle two tensor factors in the 4-fold tensor product.

- Then $\underline{\mathrm{end}}^r(A) = A \bullet A^!$, where $A^!$ is the Koszul dual to A (take the free algebra on dual vector space to A_1, quotient by the perpendicular space to the relations of A), and $\underline{\mathrm{end}}^l(A) = \underline{\mathrm{end}}^r(A^!)$.

Manin's universal quantum group $\underline{\mathrm{aut}}^r(A)$ is the Hopf envelope $\mathcal{H}(\underline{\mathrm{end}}^r(A))$ of $\underline{\mathrm{end}}^r(A)$.

Quantum-symmetric equivalence

- Two Hopf algebras H_1 and H_2 are said to be **Morita–Takeuchi equivalent** if $\text{comod}(H_1) \cong \text{comod}(H_2)$ as tensor categories.
- We say that two graded algebras A_1 and A_2 are **weakly quantum-symmetrically equivalent** if $\underline{\mathrm{aut}}^r(A_1)$ is Morita–Takeuchi equivalent to $\underline{\mathrm{aut}}^r(A_2)$, and **quantum-symmetically equivalent** if additionally the equivalence sends A_1 to A_2.
- Main question: what properties of A are shared by its quantum-symmetric equivalence class?

Morita–Takeuchi theory

- A left Galois object for a Hopf algebra H is a left H-comodule algebra A such that the composition

$$A \otimes A \xrightarrow{\rho \otimes 1_A} H \otimes A \otimes A \xrightarrow{1_H \otimes m_A} H \otimes A$$

is a linear isomorphism. Right H-Galois objects are defined similarly.

- An H_1-H_2 bi-Galois object is a bicomodule algebra which is a left H_1 Galois object and a right H_2 bi-Galois object.

Theorem (Schauenburg)

Two Hopf algebras H_1 and H_2 are Morita–Takeuchi equivalent if and only if there is a nonzero bi-Galois object between them.

Morita–Takeuchi theory

Definition

A map $\sigma : H \otimes H \to \Bbbk$ is a **2-cocycle** if it is invertible in the convolution algebra, and satisfies

$$\sum \sigma(x_1, y_1)\sigma(x_2y_2, z) = \sum \sigma(y_1, z_1)\sigma(x, y_2z_2) \quad \text{and}$$
$$\sigma(x, 1) = \sigma(1, x) = \varepsilon(x).$$

For a 2-cocycle, there is a way of twisting the multiplication of H giving a Hopf algebra H^σ, which is Morita–Takeuchi equivalent to H.

An H_1-H_2 bi-Galois object A gives a 2-cocycle if and only if it is **cleft**, that is, if there are isomorphisms $A \cong H_1$ as left H_1-comodule algebras and $A \cong H_2$ as right H_2-comodule algebras.

Zhang twists and quantum-symmetric equivalence

- Recall that if we follow the philosophy of noncommutative algebraic geometry proposed by Artin and Zhang, the "noncommutative projective scheme" of an algebra A is an invariant of the graded module category of A.
- In other words, from this perspective we only care about A up to graded Morita equivalence.

Theorem (Zhang)

Two connected graded algebras A and B are graded Morita equivalent if there is a graded automorphism $\phi : A \to A$ with $A^\phi \cong B$.

(Becomes "only if" if one replaces "graded automorphism" with "twisting system".)

Zhang twists and quantum-symmetric equivalence

Theorem (Huang–Nguyen–Ure–V.–Veerapen–Wang)

If $\phi : A \to A$ is a graded automorphism, then A and A^{ϕ} are quantum-symmetrically equivalent.

- In fact, this equivalence arises as a 2-cocycle twist. In fact, the 2-cocycles corresponding to Zhang twists are precisely the 2-cocycles that arise from 1-dimensional representations of $\underline{\mathrm{aut}}^r(A)$.
- In other words: equivalence of graded module categories, on the level of algebras, via Zhang twist from automorphism, leads to a tensor equivalence of comodule categories, on the level of their universal quantum groups.

Twisting pairs

If ϕ is a graded automorphism of A, we can get two bialgebra maps:

$$\begin{array}{ccc} A & \xrightarrow{\rho_A} & \underline{\text{end}}^l(A) \otimes A \\ \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \underline{\text{end}}^l(\phi)\otimes \text{id}} \\ A & \xrightarrow{\rho_A} & \underline{\text{end}}^l(A) \otimes A. \end{array}$$

$$\begin{array}{ccc} A^! & \xrightarrow{\rho_{A^!}} & A^! \otimes \underline{\text{end}}^r(A^!) \\ \downarrow{\scriptstyle \phi^!} & & \downarrow{\scriptstyle \text{id} \otimes \underline{\text{end}}^r(\phi^!)} \\ A^! & \xrightarrow{\rho_{A^!}} & A^! \otimes \underline{\text{end}}^r(A^!). \end{array}$$

Twisting pairs

These two maps motivate our definition:

Definition

Let B be a bialgebra. A pair (ϕ_1, ϕ_2) of algebra automorphisms of B is said to be a *twisting pair* if the following conditions hold:

1. $\Delta \circ \phi_1 = (\text{id} \otimes \phi_1) \circ \Delta$ and $\Delta \circ \phi_2 = (\phi_2 \otimes \text{id}) \circ \Delta$, and
2. $\varepsilon \circ (\phi_1 \circ \phi_2) = \varepsilon$.

Lemma

If A is a quadratic algebra and $\phi : A \to A$ a graded automorphism, then $(\underline{\text{end}}^r((\phi^{-1})^!), \underline{\text{end}}^l(\phi))$ is a twisting pair for $\underline{\text{end}}^l(A)$. In fact, every *twisting pair for $\underline{\text{end}}^l(A)$ arises in this way.*

Elementary properties of twisting pairs

Lemma

If B is a bialgebra with twisting pair (ϕ_1, ϕ_2), then

1. $\phi_1 \circ \phi_2 = \phi_2 \circ \phi_1$, *and*
2. $(\phi_1 \otimes \phi_2) \circ \Delta = \Delta$.
3. ϕ_1 *is uniquely determined by* ϕ_2, *and vice versa.*

Elementary properties of twisting pairs

If ψ is a bialgebra map $B \to B$, we obtain a unique $\mathcal{H}(\psi) : \mathcal{H}(B) \to \mathcal{H}(B)$ via the universal property:

$$\begin{array}{ccc} B & \xrightarrow{i_B} & \mathcal{H}(B) \\ \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \mathcal{H}(\psi)} \\ B & \xrightarrow{i_B} & \mathcal{H}(B). \end{array}$$

Lemma

If B is a bialgebra with twisting pair (ϕ_1, ϕ_2), then $(\mathcal{H}(\phi_1), \mathcal{H}(\phi_2))$ is a twisting pair for $\mathcal{H}(B)$.

Twisting conditions

Definition

A bialgebra B satisfies the *twisting conditions* if

1. as an algebra $B = \bigoplus_{n\in\mathbb{Z}} B_n$ is $\mathbb{Z}$-graded, and
2. the comultiplication satisfies $\Delta(B_n) \subseteq B_n \otimes B_n$ for all $n \in \mathbb{Z}$.

- If A is a quadratic algebra, then $\underline{\mathrm{end}}(A)$ satisfies the twisting conditions.
- In general, if a bialgebra B satisfies the twisting conditions, then so does its Hopf envelope $\mathcal{H}(B)$ (and so in particular $\underline{\mathrm{aut}}(A)$ satisfies the twisting conditions).
- If B satisfies the twisting conditions, then any twisting pair of B preserves its grading.

Zhang twists of Hopf algebras

Proposition (HNUVVW)

Let B be a bialgebra (resp. Hopf algebra) satisfying the twisting conditions . For any graded bialgebra (resp. Hopf algebra) automorphism ϕ of B, the Zhang twist B^{ϕ} is again a bialgebra (resp. Hopf algebra) satisfying the twisting conditions.

If H is a Hopf algebra with antipode S as in proposition, the antipode of H^{ϕ} is $S^{\phi}(r) := \phi^{-|r|}(S(r))$.

Zhang twists of Hopf algebras

Theorem (HNUVVW)

Let B be a bialgebra satisfying the twisting conditions. For any graded bialgebra automorphism ψ of B, we have the following Hopf algebra isomorphism:

$$\mathcal{H}(B^{\psi}) \cong \mathcal{H}(B)^{\mathcal{H}(\psi)}.$$

Here the isomorphism is constructed via

$$\begin{array}{ccc} B^{\psi} & \xrightarrow{\;i_{B^{\psi}}\;} & \mathcal{H}\left(B^{\psi}\right) \\ & \searrow^{i_{\psi}} & \downarrow f \\ & & \mathcal{H}\left(B\right)^{\mathcal{H}(\psi)}. \end{array}$$

Zhang twists versus 2-cocycle twists

Proposition (Bichon–Neshveyev–Yamashita)

Let H be a Hopf algebra satisfying the twisting conditions. For any twisting pair (ϕ_1, ϕ_2) of H, we have the following.

1. *$\phi_1 \circ \phi_2$ is a graded Hopf automorphism of H.*
2. *The linear map $\sigma : H \otimes H \to \Bbbk$ defined by*

$$\sigma(x, y) = \varepsilon(x)\varepsilon(\phi_2^{|x|}(y)),$$

   is a 2-cocycle, whose convolution inverse σ^{-1} is given by

$$\sigma^{-1}(x, y) = \varepsilon(x)\varepsilon(\phi_1^{|x|}(y)).$$

3. *The 2-cocycle twist $H^\sigma \cong H^{\phi_1 \circ \phi_2}$.*

As a consequence, H and $H^{\phi_1 \circ \phi_2}$ are Morita-Takeuchi equivalent.

Zhang twists versus 2-cocycle twists

Corollary (HNUVVW)

Let B be a bialgebra satisfying the twisting conditions. For any twisting pair (ϕ_1, ϕ_2) of B, there is a unique twisting pair $(\mathcal{H}(\phi_1), \mathcal{H}(\phi_2))$ of the Hopf envelope $\mathcal{H}(B)$ extending (ϕ_1, ϕ_2). Moreover, the 2-cocycle twist $\mathcal{H}(B)^\sigma$, with the 2-cocycle $\sigma : \mathcal{H}(B) \otimes \mathcal{H}(B) \to \Bbbk$ given by

$$\sigma(x, y) = \varepsilon(x)\varepsilon(\mathcal{H}(\phi_2)^{|x|}(y)),$$

is the right Zhang twist $\mathcal{H}(B)^{\mathcal{H}(\phi_1 \circ \phi_2)}$.

Manin's quantum groups

Putting this all together for $\underline{\mathrm{aut}}^l(A)$:

- A a graded algebra, $\phi : A \to A$ a graded algebra automorphism.
- Then $\underline{\mathrm{end}}^l(A)$ satisfies the twisting conditions, and $(\underline{\mathrm{end}}^r((\phi^{-1})^!), \underline{\mathrm{end}}^l(\phi))$ is a twisting pair for $\underline{\mathrm{end}}^l(A)$.
- This extends to a twisting pair $(\mathcal{H}(\underline{\mathrm{end}}^r((\phi^{-1})^!)), \mathcal{H}(\underline{\mathrm{end}}^l(\phi)))$ for $\mathcal{H}(\underline{\mathrm{end}}^l(A)) \cong \underline{\mathrm{aut}}^l(A)$.

Manin's quantum groups

Then we can compute

$$\begin{aligned}
\underline{\mathrm{aut}}^l(A^\phi) &\cong \mathcal{H}(\underline{\mathrm{end}}^l(A^\phi)) \\
&\cong \mathcal{H}((A^\phi)^! \bullet (A^\phi)) \\
&\cong \mathcal{H}((A^!)^{(\phi^{-1})^!} \bullet A^\phi) \\
&\cong \mathcal{H}((A^! \bullet A)^{\underline{\mathrm{end}}^l(\phi)\circ\underline{\mathrm{end}}^r((\phi^{-1})^!)}) \\
&\cong \mathcal{H}(\underline{\mathrm{end}}(A)^{\underline{\mathrm{end}}^l(\phi)\circ\underline{\mathrm{end}}^r((\phi^{-1})^!)}) \\
&\cong \mathcal{H}(\underline{\mathrm{end}}(A))^{\underline{\mathrm{aut}}^l(\phi)\circ\underline{\mathrm{aut}}^r((\phi^{-1})^!)} \\
&\cong \underline{\mathrm{aut}}(A)^\sigma.
\end{aligned}$$

Artin–Schelter regular algebras

We call a connected graded algebra A **Artin–Schelter regular** (AS-regular) if

(AS1) A has finite global dimension d, and

(AS2) there is some integer l so that

$$\underline{\mathrm{Ext}}^i_A(\mathbb{k}, A) = \begin{cases} \mathbb{k}(l) & \text{if } i = d \\ 0 & \text{if } i \neq d, \end{cases}$$

where $\mathbb{k}$ is the trivial module $A/A_{\geq 1}$.

Idea: AS-regular algebras play the role of coordinate rings of "noncommutative $\mathbb{P}^n$," in other words, they are noncommutative analogs of polynomial rings. Hence, the study of AS-regular algebras is a foundational problem in noncommutative projective algebraic geometry.

Quantum-symmetric invariants

Theorem (Raedschelders–Van den Bergh)

Suppose A and B are AS-regular of the same dimension. Then they are weakly quantum-symmetrically equivalent.

Quantum-symmetric invariants

Theorem (HNUVVW)

Let A be a Noetherian connected graded algebra, and let B be quantum-symmetrically equivalent to A.

- *If A is AS-regular, then B is AS-regular;*
- *If A is N-Koszul, then B is N-Koszul;*
- $\mathrm{gldim}(A) = \mathrm{gldim}(B)$*;*
- $\mathrm{ASReg}(A) = \mathrm{ASReg}(B)$, *where* $\mathrm{ASReg}(-)$ *is numerical AS-regularity, defined by Kirkman–Won–Zhang.*

Special cases of result (1) have appeared previously in the literature: Chan–Kirkman–Walton–Zhang have proven that 2-cocycle twisting for semisimple finite-dimensional Hopf coactions preserves AS-regularity and Chirvasitu–Smith have proven that 2-cocycle twisting for finite-dimensional Hopf coactions preserves AS-regularity.

Quantum-symmetric invariants

Idea of the proof:

- Main tool: the relative module category consisting of graded modules for A which are comodules for $\underline{\text{aut}}^r(A)$, in a compatible way.
- The category of graded modules of A is NOT preserved under a Morita–Takeuchi equivalence of universal quantum groups.
- E.g., by Radschaelders–Van den Bergh, an two AS-regular algebras A and B of the same dimension are quantum-symmetrically equivalent, but it is known that they can have radically different categories of graded modules (e.g. their point schemes, which classify the point modules, don't have to be isomorphic).
- But quantum-symmetric equivalence DOES preserve the relative module category.

Quantum symmetries of Artin–Schelter regular algebras

Theorem (HNUVVW)

Let A be an N-Koszul connected graded algebra. If A is AS-regular, and B is quantum-symmetrically equivalent to A via a 2-cocycle twist, then B is AS-regular.

Quantum-symmetric invariants

Idea of the proof:

- Compared to the previous theorem: can drop the Noetherian assumption on A, but add N-Koszul assumption and the assumption that the quantum-symmetric equivalence arises as a 2-cocycle twist; we can replace the computation of Ext by the condition that the Koszul dual is Frobenius, using a result of Lu–Palmieri–Wu–Zhang which states that an N-Koszul algebra is AS-regular if and only if its Koszul dual is Frobenius.
- The Koszul dual $A^!$ is also a comodule algebra for $\underline{\mathrm{aut}}^r(A)$, and the Frobenius form $A^! \otimes A^! \to \Bbbk$ is H-colinear.
- Then we show that ${}_{\sigma^{-1}}(A^!) \cong (A_\sigma)^!$, and that the Frobenius property is preserved under monoidal equivalence.

Quantum-symmetric invariants

Combining our results with the Raedschelders–Van den Bergh theorem, we describe the (various) quantum-symmetric equivalence classes of any Koszul Noetherian AS-regular algebra A of dimension d:

- the quantum-symmetric equivalence class of A, where the Morita–Takeuchi equivalence is given by a 2-cocycle coming from a 1-dimensional representation: Zhang twists of A;
- the quantum-symmetric equivalence class of A, where the Morita–Takeuchi equivalence is given by a 2-cocycle: Koszul AS-regular algebras of dimension d with the same Hilbert series as A;
- the quantum-symmetric equivalence class of A: contained in the set of Koszul AS-regular algebras of dimension d;
- the weak quantum-symmetric equivalence class of A: contains all Koszul AS-regular algebras of dimension d.

Quantum symmetries of superpotential algebras

- A **preregular form** is an m-linear form f on a vector space V such that
  1. f is nondegenerate, and
  2. there exists $\mathbb{P} \in \mathrm{GL}(V)$ with

$$f(v_1, \dots, v_m) = f(\mathbb{P}(v_m), v_1, \dots, v_{m-1}) \;\forall\; v_1, \dots, v_m \in V.$$

- The **superpotential algebra** associated to f and an integer N, denoted $A(f, N)$, is the free algebra generated by a basis $\{x_1, \dots, x_n\}$ of V, quotient by relations

$$\sum_{1 \le j_1, \dots, j_N \le n} f_{i_1 \dots i_{m-N} j_1 \dots j_N} x_{j_1} \dots x_{j_N} = 0$$

for all $1 \le i_1, \dots, i_{m-N} \le n$, where we denote $f_{i_1 \dots i_m} := f(x_{i_1}, \dots, x_{i_m})$.

Quantum symmetries of superpotential algebras

Theorem (Dubois-Violette)

Suppose A is Koszul AS-regular algebra, then there exist f and N such that $A \cong A(f, N)$.

Cogroupoids

A $\Bbbk$-cocategory $\mathcal{C}$ consists of the following data:

1. A set of objects $\text{ob}(\mathcal{C})$.
2. For any $X, Y \in \text{ob}(\mathcal{C})$, a $\Bbbk$-algebra $\mathcal{C}(X, Y)$.
3. For any $X, Y, Z \in \text{ob}(\mathcal{C})$, $\Bbbk$-algebra homomorphisms

$$\Delta_{XY}^{Z} : \mathcal{C}(X, Y) \to \mathcal{C}(X, Z) \otimes \mathcal{C}(Z, Y)$$

and

$$\epsilon_X : \mathcal{C}(X, X) \to \Bbbk$$

such that standard coassociativity and counit diagrams are satisfied.

Cogroupoids

A $\Bbbk$-cocategory is a $\Bbbk$-**cogroupoid** if it is then also equipped with linear maps $S_{X,Y} : \mathcal{C}(X,Y) \to \mathcal{C}(Y,X)$ for all X and Y such that

$$\begin{array}{ccccc}
\mathcal{C}(Y,X) & \xleftarrow[u]{} & \Bbbk & \xleftarrow[\epsilon_X]{} & \mathcal{C}(X,X) \\
\uparrow{\scriptstyle m} & & & & \downarrow{\scriptstyle \Delta^Y_{X,X}} \\
\mathcal{C}(Y,X) \otimes \mathcal{C}(Y,X) & & \xleftarrow[S_{X,Y} \otimes \mathrm{id}]{} & & \mathcal{C}(X,Y) \otimes \mathcal{C}(Y,X)
\end{array}$$

and a similar diagram with S on the right commute.

Cogroupoids

Theorem (Bichon 20014)

Two Hopf algebras H_1 and H_2 satisfy $\mathrm{comod}(H_1) \cong \mathrm{comod}(H_2)$ *if and only if there exists a cogroupoid $\mathcal{C}$ with $\mathcal{C}(X,X) = H_1$, $\mathcal{C}(Y,Y) = H_2$, and $\mathcal{C}(X,Y) \neq 0$.*

- Clear: in a cogroupoid, each $\mathcal{C}(X,X)$ is a Hopf algebra.
- Recall that a left Galois object for a Hopf algebra H is a left H-comodule algebra A such that the composition

$$A \otimes A \xrightarrow{\rho \otimes 1_A} H \otimes A \otimes A \xrightarrow{1_H \otimes m_A} H \otimes A$$

is a linear isomorphism. Right H-Galois objects are defined similarly.

Cogroupoids

If $\mathcal{C}$ is a cogroupoid and $\mathcal{C}(X, Y)$ is nonzero, then it is a $\mathcal{C}(X, X) - \mathcal{C}(Y, Y)$ bi-Galois object– that is, a bicomodule algebra which is a left Galois object for $\mathcal{C}(X, X)$ and a right Galois object for $\mathcal{C}(Y, Y)$. Then one direction of Bichon's theorem follows from a classic theorem of Schauenberg:

Theorem (Schauenberg 1996)

Two Hopf algebras H_1 and H_2 are Morita–Takeuchi equivalent if and only if there is a bi-Galois object between them.

A cogroupoid associated to preregular forms

Let e and f be m-preregular forms on vector spaces V and W, respectively, of dimensions k and l. We define $\mathcal{GL}_m(e,f)$ to be the $\Bbbk$-algebra with generators

$$\mathbb{A} = (a_{ij})_{\substack{1\le i\le k,\\ 1\le j\le l}} \qquad \mathbb{B} = (b_{ij})_{\substack{1\le i\le l,\\ 1\le j\le k}} \qquad D^{\pm 1},$$

subject to the relations

$$\sum_{1\le i_1,\dots,i_m\le k} e_{i_1\cdots i_m} a_{i_1 j_1}\cdots a_{i_m j_m} = f_{j_1\cdots j_m} D,$$
$$\sum_{1\le i_1,\dots,i_m\le l} f_{i_1\cdots i_m} b_{i_m j_m}\cdots b_{i_1 j_1} = e_{j_1\cdots j_m} D^{-1},$$
$$DD^{-1} = D^{-1}D = 1,$$
$$\mathbb{A}\mathbb{B} = \mathbb{I}_{k\times k}.$$

A cogroupoid associated to preregular forms

For each integer m, we now construct a cogroupoid $\mathcal{GL}_m$, where objects are m-preregular forms, and $\mathcal{GL}_m(e, f)$ is defined as the algebra above. The structure maps

$$\Delta = \Delta^f_{e,g} : \mathcal{GL}_m(e, g) \to \mathcal{GL}_m(e, f) \otimes \mathcal{GL}_m(f, g)$$

are now defined via

$$\Delta(a^{e,g}_{ij}) = \sum_{k=1}^{q} a^{e,f}_{ik} \otimes a^{f,g}_{kj},$$

$$\Delta(b^{e,g}_{ji}) = \sum_{k=1}^{q} b^{e,f}_{ki} \otimes b^{f,g}_{jk},$$

$$\Delta((D^{e,g})^{\pm 1}) = (D^{e,f})^{\pm 1} \otimes (D^{f,g})^{\pm 1},$$

and

$$\varepsilon_e : \mathcal{GL}_m(e) \to \Bbbk$$

such that $\varepsilon_e(a^{e,e}_{ij}) = \varepsilon_e(b^{e,e}_{ji}) = \delta_{ij}$, and $\varepsilon_e((D^{e,e})^{\pm 1}) = 1$.

A cogroupoid associated to preregular forms

Additionally, we define

$$S_{e,f} : \mathcal{GL}_m(e,f) \to \mathcal{GL}_m(f,e)^{op}$$

by the formulas

$$S_{e,f}\left(\mathbb{A}^{e,f}\right) = \mathbb{B}^{f,e},$$
$$S_{e,f}\left(\mathbb{B}^{e,f}\right) = \left(D^{f,e}\right)^{-1}\mathbb{Q}^{-1}\,\mathbb{A}^{f,e}\,\mathbb{P}\,D^{f,e},$$
$$S_{e,f}\left(\left(D^{e,f}\right)^{\pm 1}\right) = \left(D^{f,e}\right)^{\mp 1}.$$

One can check that $\mathcal{GL}_m(e,e)$ is the universal quantum group that preserves e, studied by Dubois-Violette–Launer, Bichon–Dubois-Violette, Chirvasitu–Walton–Wang.

A cogroupoid associated to preregular forms

Theorem (HNUVVW)

$\mathcal{GL}_m$ is a cogroupoid, for any m.

A cogroupoid associated to preregular forms

Proposition (HNUVVW)

For any 2-preregular forms e and f, $\mathcal{GL}_2(e, f)$ is nonzero.

Upshot: given any two superpotential algebras from 2-preregular forms, there is a Morita–Takeuchi equivalence between Hopf algebras coacting on them which takes one to the other.

A cogroupoid associated to preregular forms

This recovers two previously known results as corollaries:

Corollary

All superpotential algebras coming from 2-preregular forms are AS-regular.

This was previously known due to James Zhang's classification of dimension 2 AS-regular algebras.

Corollary

All dimension 2 AS-regular algebras are quantum-symmetrically equivalent.

Recall that this is a special case of the result of Radschaelders–Van den Bergh (their result is for arbitrary dimension).

Proof sketch

First step: reduction to the case where the dimensions of the vector spaces for e and f are the same.

- For arbitrary $\mathbb{F} \in \mathsf{GL}(W)$ with corresponding preregular form f, set

$$\mathbb{E} = \mathbb{E}_q := \begin{pmatrix} 0 & 1 \\ -q^{-1} & 0 \end{pmatrix} \in \mathsf{GL}_2(\Bbbk)$$

  such that $q^2 + \mathrm{tr}(\mathbb{F}^T \mathbb{F}^{-1}) + 1 = 0$.
- Denote e_q as the preregular form corresponding to $\mathbb{E}_q$.
- $\mathcal{GL}_2(e_q, f) \neq 0$ by a theorem of Bichon (actually, Bichon proves that a related algebra, which one can check is a quotient of $\mathcal{GL}_2(e_q, f)$ is nonzero.

Proof sketch

Reduction to the case where $\dim V = \dim W$ now follows from another result of Bichon:

Theorem (Bichon 2014)

If $\mathcal{C}(X, Z)$ and $\mathcal{C}(Z, Y)$ are nonzero, then so is $\mathcal{C}(X, Y)$.

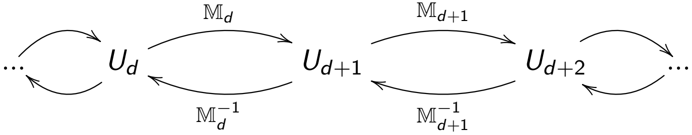
Proof sketch

Now assume $\dim V = \dim W$. To show $\mathcal{GL}_2(e,f)$ is nonzero, we construct a nonzero representation:

1. as a vector space, set $U := \bigoplus_{d\in\mathbb{Z}} U_d$, where each U_d is defined to be the 1-dimensional vector space $\Bbbk$.
2. Set $\mathbb{M}_0 := \mathbb{M}$ an arbitrary matrix in $\mathrm{GL}(V)$ and inductively define
$$\begin{cases} \mathbb{M}_{d+1} := \mathbb{E}^{-T}\mathbb{M}_d^{-T}\mathbb{F}^T & d \geq 0 \\ \mathbb{M}_{d-1} := \mathbb{E}^{-1}\mathbb{M}_d^{-T}\mathbb{F} & d \leq 0. \end{cases}$$
3. Define the action of $\mathbb{A}$ on each graded component U_d to be given by scalar multiplication $U_d \to U_{d+1}$, according to the matrix $\mathbb{M}_d$. Similarly, define the action of $\mathbb{B}$ on the graded component U_d to be given by scalar multiplication $U_d \to U_{d-1}$, given by the matrix $\mathbb{M}_{d-1}^{-1}$.
4. The action of $D^{\pm 1}$ on U_d will be defined on U_d as the multiplication by 1 from $U_d \to U_{d\pm 2}$.

Proof sketch

This gives the following diagram, where the action of $\mathbb{A}$ moves to the right, and the action of $\mathbb{B}$ moves to the left:



Defining relations of $\mathcal{GL}_2(e, f)$ are satisfied by this collection of linear maps, so it forms a nonzero representation.

Summary

This completes the proof that $\mathcal{GL}_2$ is connected. What uses does $\mathcal{GL}_n$ have for $n > 2$?

- Let e be the n-preregular form so that $A(e, 2)$ is the polynomial ring in n variables, and let f be another n-preregular form.
- Then:

$$\mathcal{GL}_n(e, f) \neq 0 \Rightarrow \operatorname{comod} \mathcal{GL}_n(e, e) \cong \operatorname{comod} \mathcal{GL}_n(f, f) \\ \Rightarrow A(f, n) \text{ is AS-regular}$$

Conclusion

Thank you for your time!